

WPI Mathematical Sciences Ph.D. General Comprehensive Exam MA 540, September 2009

Show all your work. You may quote named results. Each part is worth 10 points.

1. Suppose X and Y are random variables defined on the same probability space and having finite variances. If $c = \text{cov}(X, Y)$, what is $\text{cov}(X, E(Y|X))$?
2. Let X_1, X_2, \dots be iid random variables with mean μ and finite variance. Show that $Y_n = 2[n(n+1)]^{-1} \sum_{j=1}^n jX_j$ converges in probability to μ .
3. Let $\{A_n\}$ be a sequence of events on the probability space (Ω, \mathcal{F}, P) .
 - (a) Define $\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} A_i$. Show that $\liminf_n A_n$ equals the event consisting of all outcomes that occur in all but a finite number of the A_n .
 - (b) Define $\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Give an explanation of this event analogous to that given in part (a) for $\liminf_n A_n$, and prove your contention.
4. For each of the criteria listed, construct a sequence of random variables $\{X_n, n \geq 1\}$ satisfying the criteria. Be sure to prove your results.
 - (a) Converges in probability but not with probability 1.
 - (b) Converges in probability but not in mean square.
5. Bootstrap samples are obtained by sampling randomly with replacement from an original data set. If the original data set has n observations, a bootstrap sample also consists of n observations.
 - (a) What is the chance that any given observation, say observation 1, is in a bootstrap sample?
 - (b) What is the limiting value of this probability as $n \rightarrow \infty$?
 - (c) What is the expected number of distinct observations in a bootstrap sample?
6. Let X_1, X_2, \dots be iid with exponential distribution with mean θ . Let $T_0 = 0$ and $T_n = \sum_{i=0}^n X_i$. Fix $t > 0$ and let Y be the maximum integer $n \geq 0$ such that $T_n \leq t$. Show that Y has a Poisson distribution with mean t/θ .